Sub-scientific mathematics: undercurrents and missing links in the mathematical technology of the Hellenistic and Roman world

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Article written in 1990 for
***Aufstieg und Niedergang der römischen Welt* II, vol. 37,4**
which is still unpublished in May 2007

Manuscript

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I. Ancient mathematics: theory or technology?

Greek mathematics – to anybody who possesses the faintest idea of the history of mathematics, this means something in the style of Euclid's *Elements*, of Archimedes, of Apollonios, of Diophantos and of Pappos. That is, »Greek mathematics« (or, we might generalize, referring to the »mathematics of Classical Antiquity«, or be precise and speak of »Hellenistic mathematics«) is a field of knowledge concerned with theoretical understanding of abstract entities. Those whose ideas are less faint may know about the Heronian corpus, about Ptolemy, and about similar applications of the abstractions to describe material reality – »the more physical of the branches of mathematics«, to speak with Aristotle[1]. Still, *theory* retains the primary role, and the rest remains derivation – »subordinate«, if we stick to Aristotelian parlance[2]. Neo-Pythagorean works like those of Nicomachos, of course, change nothing in this respect, and Hero constitutes a minor exception, the very distinctiveness of which seems to confirm the global rule.

As it has been pointed out by G. J. TOOMER[3], this image of Greek mathematics is produced by a somewhat distorting lens: The ideals of the schoolmen of late Antiquity and early Medieval Byzantium. *They* decided which manuscripts were to be copied and be preserved with sufficient care. The effect of this process of spontaneous censorship is revealed by the character of those works which are only known via Arabic translations[4]. Among the works which were translated into Arabic around A.D. 800 and thus still available in Greek at that date but often only in one defective manuscript and not in the late Middle Ages are:

- Euclid's treatise of the division of figures.
- A number of presumably Archimedean works, dealing *inter alia* with

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[1] *Physica* 194a7; transl. R. P. HARDIE and R. K. GAYE 1930.

[2] E.g., *Posterior Analytics* 75b14-16; transl. G. R. G. MURE 1928.

[3] TOOMER 1984.

[4] In several cases, where the Arabic translations have also been lost, their content has been incorporated or paraphrased in surviving texts, or they have been translated into Latin in the twelfth century.

the construction of the regular heptagon and the construction of water-clocks.

- Menelaos' *Spherics* and his treatise on the mathematics of specific gravities.
- Ptolemy's *Planisphaerium* and *Optics*.
- Books 5-7 of Apollonios' *Conics* and his *On cutting off a ratio*.
- Pappos' *Commentary to Book X of Euclid's Elements*, the passage of his *Collection* dealing with constructions with fixed compass opening and part of Book VIII on mechanics.
- Part of Diophantos' *Arithmetica*[5].

Even though there is no obvious system in this list, it suggests that works which did not agree with the canon of »compass and ruler«, which were too sophisticated, or which belonged to the Aristotelian category of »subalternate sciences« (optics, mechanics, spherics) were more likely to be neglected than others.

It can be easily argued that this bias corresponds to the attitudes expressed by a multitude of later Hellenistic and late Ancient authors from Plutarch to Proclos[6]. To them, mathematics was, in itself, either a way to gain higher insight or, more modestly, a propaedeutic paradigm by which the ability to gain insight was trained – or it was a hermeneutic aid, necessary for the interpretation of Plato and Aristotle. Hermeneutic assistance apart, however, these attitudes are close to those expressed by Plato and Aristotle in the fourth century B.C., in whose vicinity mathematicians like Theaetetos and Eudoxos made their work; they correspond fairly well to the style of the major mathematicians[7], and even to those of the lost works which exist in genuine Arabic translation; they are not contradicted by mathematically competent commentators and compilators from Geminos to Pappos, Theon and Eutocios. Byzantine scholars, furthermore, were not too strict in their criteria, as demonstrated,

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[5] For detailed information, see the relevant articles in DSB and (as far as more recent discoveries are concerned) TOOMER 1984.

[6] See, e.g., the oft-quoted passage in *Tabletalk* VIII.2 (ed., transl. MINAR et al 1969: 118-131), where Plutarch and his table companions discuss Plato's reasons for claiming (as they suppose he did) that »God is always doing geometry« .

[7] Ptolemy, so it seems, did not even regard the high-level logistics of astronomical computations as a *mathematical* activity, as pointed out by OLAF PEDERSEN (1974: 32-34).

e.g., by their compilation of the Heronian *Geometrica*. All in all, then, the lens of the late schoolmen can be seen to have been somewhat distorting; but it certainly did not change the total picture, nor *a fortiori* produce an illusion. Greek and Hellenistic mathematics, *in its culturally and quantitatively dominating form*, *was* theoretical and concerned with abstract entities – »pure«, we would say. What is more: Even works which according to their contents were »applied« (dealing with physical or astronomical reality like, e.g., Archimedean statics or Autolycos' spherics) tended to be *formally pure*, demonstrating thus their dependence on the abstract fundament.

To us, this may seem the natural order of things, heirs to the Hellenistic tradition as we are. Ever since the French École Polytechnique was established in 1794, engineers have been taught their applied mathematics according to the same model. Seen in the context of the Ancient world, however, the pattern of Greek and Hellenistic mathematics is outstanding. The Romans only accepted it halfheartedly if at all. This is clearly stated by Cicero in the *Tusculan Disputations*[8]: »With the Greeks geometry was regarded with the utmost respect, and consequently none was held in greater honour than mathematicians, but we Romans have restricted this art to the practical purposes of measuring and reckoning«. A demonstration *ad oculos* is provided by Quintilian in the passage of *De institutione oratoria* where the relevance of geometry is explained[9]: Firstly, the term geometry is taken to include plain numerical computation; secondly, the main aim of teaching the subject is to avoid elementary blunders in basic practical numerical and field-surveying calculations. *Roman* mathematics at its best, on this evidence, was not Euclid, nor even Hero's deliberate adaptation of theoretical results for use in practice; it is adequately represented by the agrimensors' secondary adoption of Heronian and similar Alexandrian material.

The Greek and Hellenistic pattern is also radically different from *earlier* mathematical traditions. Babylonian and Egyptian mathematics (the only early traditions which are clearly documented and clearly dated[10])

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[8] I, ii; transl. KING 1971: 7.

[9] I, x, 34ff; ed., transl. H. E. BUTLER 1969: IV, 176ff.

[10] The Indian geometrical tradition as documented in the *s | ulba sūtras* may antedate the rise of Greek mathematics; but in their present form, the *sūtras* may be roughly contemporary with the early Greek mathematicians (c. 500 to c. 300 B.C.) (BAG 1979: 4-6). Chinese mathematics is only documented from the Han era onwards

originated as *technologies*, as techniques for accounting, for field measurement, and for the planning of provisions for workers and soldiers. In the long run, Babylonian mathematics certainly did not stick to this »applied« character: many of its characteristic problems, indeed whole disciplines, are definitely non-utilitarian. But however »pure« the contents, the *form* remained »applied«[11] (on a more modest scale, the same can be stated of Egyptian mathematics). Even when the mathematics of the scribal cultures was non-utilitarian (»pure«), it was never *theoretical*, neither in Greek nor in modern sense[12]. In mathematics (as elsewhere), Greek culture created something radically new – something which was then institutionalized and ripened in the early Hellenistic era, in particular around Alexandria, and conserved and canonized by the Hellenistic schoolmen throughout the Roman period.

Apart from their »pure« outgrowth, Babylonian and Egyptian mathematics had corresponded to obvious social needs of a practical nature. Evidently, these needs were not abolished at the birth of Thales nor through the Macedonian conquest. Nor were they covered by the sparse works applying theoretical results to more practical problems, which, in fact, were either concentrated within select areas (predominantly other sciences like astronomy) or attempts to improve upon the bad methods used by rank-

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(MARTZLOFF 1988: 110ff).

[11] One striking example of »pure commercial arithmetic« is the following problem: »I have bought 770 sila [¸liter] oil. From what was bought for 1 šeqel of silver I cut away 4 sila each time. I saw 40 šeqel of silver as profit. How much did I buy and how much did I sell [for each šeqel]?« (my translation from the transliteration in BRUINS & RUTTEN 1961: 82; the meaning is that if r sila are bought per šeqel, $r-4$ sila are sold at that price). Thanks to the fiction of a merchant who buys and sells at prices he does not know, a second-degree equation is produced – something which would never happen in real practical computation within the Babylonian horizon.

Like the vast majority of Babylonian mathematical texts, this problems dates from the Old Babylonian era (c. 1900 to c. 1600 B.C.).

[12] This does not mean, as often claimed, that Babylonian mathematics was based on empirically discovered rules and on rote learning. As I have documented in a large-scale investigation of the techniques and mode of thought of Babylonian »algebra« (HØYRUP 1990), it was based on intuitively meaningful manipulation of geometrical figures. But it did not *aim at insight*, and thus was not theoretical in the Greek sense. Nor was it organized in a explicitly formulated coherent conceptual structure, whence it cannot be considered theoretical in a modern sense.

and-file practitioners[13]. Already from first principles we can thus be sure that practical arithmetical and geometrical computation – the two fundamental mathematical technologies – lived on throughout the Classical age; *by name* we also know about logistics and geodesics from Geminos[14] as well as Aristotle and Plato and a number of commentators, and in the *Corpus iuris civilis*, *calculatores* are mentioned a few times on a par with librarians, nomenclators (slaves telling names of persons met or of fellow slaves to their master), stenographers, stage-players and other performers of practical arts[15]. Unlike its scribal predecessor traditions, however, this practical computation and its carriers had stopped being culturally productive; to a large extent their existence was not even recognized by the culturally productive stratum, and we are thus told virtually nothing about the actual ways and tasks of these lowly people, beyond, e.g., their use of ropes and rulers[16] and of concrete (»sensible«), not abstract numbers.

Some supplementary evidence comes from administrative Greco-Egyptian papyri, from descriptions of and materials for elementary teaching[17], from pictorial representations of calculators manipulating calculi on an abacus, and from surviving specimens of this device. On the whole, however, material of Classical provenience tells us fairly little about

[13] In the introduction to his treatise on the *Dioptra* (ed., transl. SCHÖNE 1903: 188ff), Hero explains his aim to be to correct earlier errors, those committed to writing as well as those committed in actual siege warfare. One of the blunders which Quintilian wants to eliminate through the teaching of geometry is the measure of areas by means of the circumference of figures, accepted according to him by almost everybody (*De institutione oratoria* I, x, 39; ed., transl. BUTLER 1969: 178f) and apparently also used throughout Classical Antiquity by practitioners. Admittedly, the evidence cited for this by EVA SACHS (1917: 174) is weak when taken in itself – Thukydides (*History of the Peloponnesian War* VI, i, 2; ed., transl. VOILQUIN 1950: II, 69) does not argue from the time of circumnavigation of Sicily for its area but for the difficulty of the Athenian military adventure – but cf. corroborating evidence cited below, section VII.

[14] Fragment on the mathematical sciences; ed., transl. AUJAC 1975: 115f.

[15] *Digest* XXVII, i,15,§5; XXXVIII,i,7,§5; L,xiii,1,§6. Cf. MARROU 1956: 431 and KINSEY 1979.

[16] Geminos, fragment on mathematics, ed., transl. AUJAC 1975: 115f.

[17] The most important specimen being the compendium in GUERAUD & JOUGUET (eds, transls) 1938. Such material for primary education, of course, carries little information on the more specific ways of professional calculators.

the basic mathematical technologies of the Hellenistic and Roman world. In particular, it does not inform us whether (or to which extent) they were ultimately derived from the theoretical mathematics of the age, indigenously but autonomously developed, or borrowed from older neighbouring cultures.

II. Sub-scientific mathematics

Left with Greek and Roman sources alone, we would thus have to content ourselves with the observation that practical arithmetic and geometry existed and were distinguished sharply from theoretical mathematics. Happily, however, we are not left with Greek and Roman sources. Earlier and later mathematical cultures have given us their own documents, which happen to make new sense of scattered and otherwise unapparent evidence in the Classical sources. Before discussing this directly we shall, however, introduce some general observations on the different varieties of mathematical activity in the pre-Modern world.

A passage from Aristotle's *Metaphysics* – dealing not with mathematics but with productive arts and theoretical knowledge in general – may introduce the problem:

> At first he who invented any art whatever that went beyond the common perceptions of man was naturally admired by men, not only because there was something useful in the inventions, but because he was thought wise and superior to the rest. But as more arts were invented, and some were directed to the necessities of life, others to recreation, the inventors of the latter were naturally always regarded as wiser than the inventors of the former, because their branches of knowledge did not aim at utility. Hence when all such inventions were already established, the sciences which do not aim at giving pleasure or at the necessities of life were discovered, [...]
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> So [...], the theoretical kinds of knowledge [are thought] to be more the nature of Wisdom than the productive.[18]

First of all, this introduces the distinction between productive and theoretical knowledge and establishes precisely that mutual ranking which

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[18] $981^{b}14$-$982^{a}1$ – ed., transl. ROSS 1928.

was noted above for the case of mathematical disciplines. Secondly it presupposes that the two kinds of knowledge are carried by different (groups of) persons: logistics and geodesics are not supposed to be performed by arithmeticians and geometers, the theoretical mathematicians. Thirdly, even productive knowledge is pointed to as going »beyond the common perceptions of man«, i.e., to be *specialists' knowledge.*

We might also speak of *craft knowledge*. Specialists in the practical arts, indeed, belong to different crafts, whose members were until the onset of the Modern era (in most cases, until the present century) trained within the profession, either as apprentices or, in exceptional cases (Babylonian and Egyptian scribe schools, the Abacus school of Late Medieval and Renaissance Italy), in specific schools. The diffusion of knowledge from the theoretical sciences (after the emergence of these during Classical Antiquity) was slow and random, not systematized as in Modern engineering schools, where teachers who have themselves been trained at a university teach future engineers their physics and mathematics, thus ensuring the diffusion of relevant results within a single generation. Minor exceptions disregarded, the knowledge of practical specialists was thus autonomous, and not to be understood as »applied science«[19]. At the same time, the knowledge of a craft constitutes, in the likeness of a scientific discipline, an *organized body of knowledge* and not a mere heap of random and disconnected rules; but cognitive coherence is no primary aim in itself but only a by-product of the practical coherence of the activity of the craft, whose members (e.g., geometrical practitioners) will often attend to a number of different practical tasks united by the fact that they can be dealt with by means of the tools and specific methods of the craft[20]. In order

[19] Even today, it is true, the idea of technology as »applied science« should be handled with uttermost care, as it has been established by historians of technology in recent decades. Still, the diffusion of scientific knowledge through the network of teachers and the deliberate search for relevant knowledge makes the »application of science« *one* important aspect of modern technologies.

[20] In the case of professions carried by a scribal or similar school, a supplementary source for cognitive coherence is the systematizing dynamics of the school – a fact to which we shall return below. The aggregate outcome of these two drives for secondary cognitive coherence – the practical coherence of professional tasks as reflected in schoolmasters' ideals – is that casuistic organization which characterizes not only Babylonian (and Egyptian) mathematical texts but also »Hammurapi's Law« and the Babylonian omen literature.

to emphasize both the organized character of this kind of specialists' knowledge going »beyond the common perceptions of man« and the distinct character of this organization, I have suggested the term »sub-scientific knowledge«[21].

In the following, we shall concentrate on sub-scientific *mathematics*, even though the concept has wider currency. Like Babylonian scribal mathematics, sub-scientific mathematics in general possesses a »pure«, i.e., non-utilitarian level, which can be regarded as its »cultural superstructure«. None the less, the *raison-d'être* of a body of practitioners' knowledge remains its adequacy with respect to the practical tasks of the professional group in question. The utilitarian basis of a body of sub-scientific mathematical knowledge is thus determined by problems, and its characteristic methods and conceptual tools have been developed with the aim of coping with these problems. To this extent, the basic structure of sub-scientific mathematics is similar to the central principle of theoretical mathematics of Greek type. The key to the development of Greek mathematics, too, was the *problem*, notwithstanding its usual textual presentation in the form of axioms, theorems, etc.

Firstly, the importance of the three »classical problems« is well-known: *viz.* doubling the cube, trisecting the angle, squaring the circle. When these were first approached as specific geometrical problems, presumably in the late fifth century B.C., no theoretically acceptable methods were at hand allowing solution; from Hippocrates of Chios and Archytas onwards, incessant attempts were made to solve them by means of methods more satisfactory than those found by earlier workers[22]. But a case of even greater consequence is provided by the theory of irrationals. The first discovery of irrationals – itself a result of theoretical investigation – highlighted problems which could not have been formulated at the level of common sense (as could the »classical problems): *how to construct* according to a general scheme lines which are not commensurate with a given line (or whose squares are not commensurate with a given square); *how to classify* magnitudes with regard to commensurability; and *which are*

[21] The most thoroughly discussion of the concept will be found in HØYRUP 1990a, on which the present article draws on a number of points.

[22] See, e.g., HEATH 1921: I, 218-270, "Special Problems". The wider perspectives of the issue are dealt with extensively in KNORR 1986, which it would lead too far to discuss in detail.

the relations between different classes of irrationals? The first problem was addressed by Theodoros according to Plato's *Theatetus* 147D[23]; according to the same passage, the young Theaetetos made a (seemingly first) attempt at the second problem; *Elements* X, finally, is a partial answer to all three problems. Later on, all of them were taken up by Apollonios in work which is now lost but described by Pappos[24].

Paradoxically, the »pure« level of sub-scientific mathematics is different. It is determined not by problems but by its stock of methods and selects its problems according to their tractability by this stock at hand. To understand why, we may look at the expressions and functions of this cultural superstructure. There are two such functions, though interconnected and not always to be clearly distinguished: teaching, and the formation of professional identity and pride.

Teaching of future practitioners, evidently, aims at transmitting acquaintance with existing methods and skill in using them. This is a question of training, not of understanding or familiarizing with abstract theorems, and since the Bronze Ages the main medium for this has always been the *exercise problem* – in so far as it has not been supervised participation in genuine practice. Participation has left few detectable textual traces, while collections of exercise problems constitute our main sources for several mathematical cultures (not least Babylonian and Egyptian mathematics). The problems in question, however, are not in themselves fundamental, in the sense that they are posed because somebody needs or wants their solution – they are nothing but pretexts for the application of existing methods, and constructed so as to allow the practice of these. Problems, in other words, are *a means*, geared to the core of the subject-matter to be taught, the existing stock of methods.

The *formation of professional identity and pride* is served in particular by so-called »recreational problems«, one specimen of which we may look at:

[23] Ed., transl. FOWLER 1977.

[24] *Commentary on Book X of Euclid's Elements*, ed., transl. THOMSON & JUNGE 1930: 63f. According to the same passage in the commentary, the mature Theaetetos was responsible for the substance of *Elements* X. (Some doubts as to the identity between the conserved Arabic text and Pappos' original commentary have been raised, see BULMER-THOMAS, "Pappus of Alexandria", DSB X, 299f).

> A paterfamilias had a distance from one house of his to another of 30 leagues, and a camel which was to carry from one of the houses to the other 90 measures of grain in three turns. For each league, the camel would always eat 1 measure. Tell me, whoever is worth anything, how many measures were left.[25]

The problem is found in a Carolingian anthology of which more shall be said below. For the moment, we shall concentrate on the characteristics of the problem text itself.

Firstly, it is strikingly unrealistic in spite of its apparently daily-life subject-matter. Unless an astute trick (an intermediate stop after 20 leagues) is introduced, exactly *nothing* will be left. »Recreational problems« owe their entertainment value precisely to such grotesquerie and unexpected coincidences.

Secondly, the format is that of a riddle. No wonder that the anthology in question went together with a collection of riddles in many Medieval manuscripts, nor that Book XIV of the *Anthologia graeca* (to which we shall also return) combines »recreational problems«, riddles and oracles.

Thirdly, however, the riddle is for specialists only. As far as the present problem is concerned, this is perhaps not obvious to readers in a world where basic (and even not quite basic) numeracy is widespread. In the pre-Modern world, however, only the professional specialists would be able to follow the solution – not to speak of finding it. In the Roman world, even the majority of the generally educated would be at a loss, as was the (apparently not uncommon) orator told about by Quintilian, who »contradicts the calculation which he states in words by making an uncertain or inappropriate gesture with his fingers«[26]. Thus, by being able to solve the riddle you demonstrate (to yourself as well as to others) that you belong to the select members of the calculators' craft – that you are »worth something«.

This point may stand out more clearly in another »recreational problem«, belonging to the widespread class »purchase of a horse«:

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[25] *Propositiones ad acuendos iuvenes*, problem 52, version II, ed. FOLKERTS 1978: 74; my translation.

[26] *De institutione oratoria* I, x, 35; transl. BUTLER 1969: 177. According to the context, the inappropriate finger-reckoning gestures imply that the orator has learned by heart a result found by others, not being able to find it himself.

> Two men in possession of money found a horse which they wanted to buy; and the first said to the second that he wanted to buy it. If you give me $^1/_3$ of your money, I shall have the price of the horse. The second asked the first for $^1/_4$ of his money, and then he would equally have the price. The price of the horse and the money of each of the two is asked for.[27]

That this problem is intended for specialists will be obvious. Even in our times, few but those who remember their school algebra will know how to approach it, and even the majority of these might give up at the versions involving three or more buyers. All elements of the problem would of course be familiar to merchants of Antiquity and of the Middle Ages. The total situation, however, is as unrealistic as anything could be – already for the reason that the price of the horse can be any multiple of 11. Similarly, to combine the staple methods of commercial arithmetic in a way which solves the problem requires skill and dexterity in a world not yet in possession of symbolic algebra – and even more than ordinary skill. If you find the solution without hesitation you are really »worth something« within the community of reckoners.

This (and not plain and vaguely defined fun, as the misleading name of the genre suggests) is precisely the function of the mathematical puzzles. To a large extent, professional identity and pride consists in awareness of one's professional skill. In principle, of course, this skill is displayed in actual professional practice. But the mathematical problems presenting themselves in the everyday practice of an accountant or merchant will soon become trivial, and hence not fit for kindling anybody's vanity. Here problems like the »purchase of a horse« come to serve: more complex than everyday problems yet still looking as if they belong within the professional domain, and still solvable by current professional techniques – but only on the condition that you *are* fairly clever. Problems of this category set aside the members of the craft as particular, and particularly clever, people (whether in the opinion of others or in their professional self-esteem) – and set aside those who are able to solve the problems as craft members

[27] Leonardo Fibonacci, *Liber abaci*, ed. Boncompagni 1857: 228; my translation. The type was most popular in the Middle Ages (Leonardo gives a number of variants with three, four or five participants); but as we shall see below, it was already known to Diophantos.

par excellence. In order to do this they have to make use of the characteristic techniques of the craft. Like the problems made for teaching, they are thus constructed around the stock of existing methods – at times enlarged by specific tricks like the intermediate stop, which, once found, become a sanctioned part of that stock and of professional sub-culture in general without possessing any utilitarian function; as it shall be explained below (in note 79), a process of this kind appears to be the origin of second-degree algebra.

So far, all sub-scientific mathematical activity was treated as a uniform phenomenon. This is certainly a rough approximation, and distinctions may be introduced along many dimensions – level, degree of specialization, reckoning versus geometry. One dichotomy of importance for the understanding of the difference between the Roman world and the Bronze Age cultures is that between scribal versus non-scribal organization, between school and apprenticeship transmission of the professional tradition.

This dichotomy reveals itself (*inter alia*) in the attitude to non-utilitarian problems. The typical attitude of non-scribal reckoners is described by the mid-tenth century Damascene textbook author al-Uqlīdisī. He tells about reckoners who (when exposed to the problem of repeated doublings of unity, of type »chess-board problem«)

> strain themselves in memorizing [a procedure] and reproduce it without knowledge or scheme, [and by others who] strain themselves by a scheme in which they hesitate, make mistakes, or fall in doubt.[28]

Scribal reckoners, on the other hand (be they Babylonian scribes or Medieval clerks), will have been trained in agreement with the typical spirit of the school, according to a fixed curriculum constructed with some degree of systematic progress and involving some sort of explanation or description of principles[29].

Certain Greco-Egyptian papyri demonstrate the survival of some kind of »scribal schooling« moulded upon the traditional Egyptian pattern albeit

[28] *Book on the Chapters of Hindu Reckoning*, ed., transl. SAIDAN 1978: 337.

[29] This is precisely what is often denied concerning Babylonian and Egyptian scribal mathematics; but cf. above, note 12.

presumably in weakened form – in particular the slightly postclassical Papyrus Akhmîm[30]. Elsewhere, where no such antecedents could make their influence felt, whatever scanty evidence we have suggests the »apprenticeship model«[31], with what that implies for the character of sub-scientific mathematical knowledge (no drive toward systematization and order, etc.).

III. Traditions

As long as they have existed, crafts have transmitted their cunning from one generation of practitioners to the next, and they have borrowed (as a rule selectively) from neighbouring cultures. The same can be supposed regarding the mathematical techniques of computation, geometrical calculation, and practical-geometrical construction. But independent invention of similar techniques is a no less recurrent phenomenon, and no less to be assumed in the case of practical mathematics.

In the case of simple applied arithmetic, it is impossible to decide whether shared knowledge and similar techniques indicates diffusion from one culture to another or common response to similar practical problems. Even as complex a procedure as the division on the Medieval abacus[32] is demonstrably devised anew time and again. Shared basic arithmetical techniques do not prove the existence of connections between mathematical practitioners of different cultures. The same holds for elementary geometrical constructions and simple area calculation, including a number of »wrong«, approximate formulae which are near at hand.

This is the reason that »recreational« problems are important, not only for the understanding of the cultural sociology of the craft of reckoners but as »index fossils«. One thing is to observe that the problem of repeated doublings of unity is found in Bronze Age Babylonia, Roman Egypt, Carolingian France, and Medieval Damascus and India. This could still

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[30] Ed., transl. BAILLET 1892; see, e.g., pp. 34f and 59ff of the commentary.

[31] See, e.g., KINSEY 1979. But the evidence *is* scanty.

[32] For instance: 3390 divided by 188. Instead of counting off 188 repeatedly, you see how often you can count off 200. Answer: 16, leaving 190. But each time you have removed 200 you have taken away 12 too much, all in all $12 \cdot 16=192$; the real remainder is thus $190+192=382$, from which you may take away 200 once, leaving a true remainder of $182+12=194$, i.e., an extra 188 and a remainder of 6. 3390 divided by 188 thus gives a result of 18 and a remainder of 6.

be a non-utilitarian play occurring naturally to anybody trading in numerical computation. But when al-Uqlīdisī observes that »this is a question many people ask. Some ask about doubling one 30 times, and others ask about doubling it 64 times«[33]; when we know that the still famous »chess-board« version consists in 64 doublings, while all the other versions cited have precisely 30 doublings, there can hardly be any doubt that the motif was borrowed: nothing in the nature of numbers or geometrical series suggest the choice of 30 members, only few of the problems speak of days, and only one (late) explicitly to the days of a month[34].

Apart from odd problems, certain peculiar expressions and weird geometrical approximations can serve as index fossils. Taken together, the evidence demonstrates the existence of a number of enduring diffusion patterns, the identity of which shall be briefly mentioned in the present section of the article – more detailed information follows below[35].

One can be defined as the »Silk Road community«, the community of traders interacting in Antiquity along this combined caravan and sea route and its extensions, reaching from China to Cadiz, and encompassing in the Middle Ages at least the Mediterraneo-Islamo-Indian trade network with its offshoots. Within this whole area, recreational mathematical puzzles appear to have migrated as »camp fire riddles« for professional traders.

Evidently, we have no direct testimony of this oral mathematical culture. But from all over the area we know either problem collections or, more often, arithmetical textbooks including favourite problems. Everywhere, indeed, mathematicians behaved towards *their* oral tradition as did Apuleius, Boccaccio and others with regard to the treasure of anonymous folk tales known to them: Borrowing, pilfering, and putting in »better taste« – which last thing means in mathematics giving explicit rules and proofs, and ordering according to mathematical principles (cf. the first quotation from al-Uqlīdisī, which mentions the oral tradition and criticizes its lack of principles).

Another network of diffusion, revealing itself in a particular way to

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[33] Ed., transl. SAIDAN 1978: 337.

[34] Bhaskara II, *Līlāvatī*, ed., transl. COLEBROOKE 1819: 55.

[35] See also HØYRUP 1990a for details. A general overview of widespread problem types with references to single occurrences is given in TROPFKE/VOGEL 1980: 513-660.

speak of fractions, seems to be restricted to the Semitic-speaking area and its immediate Mediterranean contacts in Antiquity and the Middle Ages. A third network, finally, is connected to surveyors or other practical geometers; it has links backward to Old Babylonian mathematics, and certain of its characteristic ways turn up in Hellenistic and Roman sources.

Since the evidence for the existence of these networks is always indirect, it is not possible to determine to which extent they were carried by distinct professional groups within, e.g., Hellenistic and Roman society. Evidence for partial overlap of carriers will be mentioned below.

IV. »Silk road« influence in the classical world

Above, the simplest version of the »purchase of a horse« was quoted. More often, the problem involves three or more potential buyers, of which the first needs (e.g.) one third of what the others have together, the second (e.g.) one fourth of the total possessions of the others, etc. This is one of the problem types which turns up everywhere along the »Silk Road« trading network. Most of the evidence, it is true, comes from the Medieval era[36], and there is no reason to cite it in detail. But three examples can be found in Classical sources: Firstly, Diophantos' *Arithmetica* I,xxiv[37]. As always in Diophantos, it treats of pure numbers and not of money. The structure, however, is unmistakable: to find three numbers (say, A, B and C), so that $A+{}^1/_3(B+C) = B+{}^1/_4(A+C) = C+{}^1/_5(A+B)$. Apart from the abstract formulation, this problem coincides precisely (coefficients included) with another purchase found in Leonardo Fibonacci's *Liber abaci*[38]. Secondly, *Arithmetica* I,xxv, which involves four unknown numbers and the successive fractions ${}^1/_3$, ${}^1/_4$, ${}^1/_5$ and ${}^1/_6$. Thirdly and finally, a hint in passing at the characteristic clothing (»to buy in common or sell a horse«) as to something familiar occurs in Book I of Plato's *Republic*[39].

Another widespread type is the »give and take«: A says »if you give me P, I shall have m times as much as you«; B answers »but if you give

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[36] All examples mentioned by TROPFKE / VOGEL (1980: 609f) except the Chinese ones are Medieval or early Modern.

[37] Ed., transl. TANNERY 1893: I, 56-59.

[38] Ed. BONCOMPAGNI 1857: 245.

[39] 333b-c, ed., transl. SHOREY 1978: I, 332f. I am grateful to Benno Artmann for directing my attention to this passage.

me Q, I shall have n times as much as you«[40]. Even this problem is treated (in pure numbers) by Diophantos, *viz.* in *Arithmetica* I,xv. It would of course be possible, if these Diophantine problems had been quite isolated in their epoch, to claim that Diophantos was the original source and the later »recreational« versions nothing but derivations in disguise[41]. But they are not, as shown by earlier Greek as well as Chinese evidence. One Greek source is the Greco-Egyptian mathematical papyrus Michigan 620, which dates from no later than the early second century C.E.[42] Like quite a few problems from Diophantos' *Arithmetica* I, it deals with linear problems with several unknowns, and solves them by means of the αριθμός (abbreviated ς) representing the unknown number in a way reminding much of Diophantos (and even more, perhaps, of the more straightforward procedure presumably added by a scholiast in I,xviii, I,xix and I,xix, where Diophantos becomes too elegant and sophisticated); it seems to excel in the same reference to »ratio with excess« which abounds in Diophantos, and which is one of the key concepts of Euclid's *Data* (cf. below, section VI); and depending on an ambiguous restoration, one of the problems may even coincide in mathematical detail with *Arithmetica* I,xx. But in contradistinction to Diophantos and in the likeness of Medieval material of sub-scientific origin, its problems seem to deal with quantities of *drachmas*, not with pure numbers, i.e., numbers of *monads*[43].

Another interesting piece of evidence is Iamblichos' discussion of »Thymaridas flower« in his commentary to Nicomachos' *Introduction to Arithmetic*[44]. In order to show the general applicability of the rule he illustrates its strength by means of two examples which are fully in the spirit of *Arithmetica* I,xvi-xxi though actually not to be found in Diophantos. The argument presupposes that such problems were somehow considered of importance, i.e., that some group in Iamblichos' third century (and, we

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[40] See TROPFKE / VOGEL 1980: 610f.

[41] This was in fact intimated by KURT VOGEL (1954:219), before his translation of the Chinese *Nine chapters*. WOEPCKE's pure-arithmetical translation (1853: 77 nº 26) of a problem in al-Karajī which in the Arabic original deals with horse-trade (JACQUES SESIANO, private communication) seems to have called forth the misunderstanding.

[42] Published in ROBBINS 1929; cf. VOGEL 1930.

[43] VOGEL 1930: 373f.

[44] Cited from HEATH 1921: 94ff.

may assume with some confidence, in Thymaridas' fourth century B.C.) took interest in linear algebraic problems with several unknowns.

The Chinese evidence is found in Chapter VIII of the *Nine Chapters on the Mathematical Art*, the *Jiuzhang suanshu*[45], which dates no later than the early Christian era[46]. N° 10 is a precise mathematical analogue to the two-person »purchase of a horse« quoted above from Leonardo Fibonacci. N^{os} 12 and 13 are analogues of the 3- and 5-person purchases of a horse (though in variant dress[47]) in the version where each potential buyer asks the following and not everybody (a type dealt with extensively by Leonardo and related to *Arithmetica* I,xxii−xxiii).

The technique used in the *Nine Chapters* to solve these problems differs from the one used by Diophantos; it consists in a sophisticated manipulation of numerical arrays ordered in a matrix, which can hardly be imagined to be the way used by those who carried the characteristic riddles along the trade routes. In any case, the difference in method excludes both that the Chinese should have borrowed everything from an early Greek precursor of Diophantos and, vice versa, that Diophantos

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[45] Ed., transl. VOGEL 1968.

[46] VOGEL (1968: 5) cites the statement of the mathematician and commentator Liu Hui from A. D. 263, according to whom the final version of the work dates from the first century B.C. I and DU' (1987: 35) point out that the work is absent from a catalogue of books from the late first century B.C. where it should probably have been mentioned had it been in existence, while a text from c. A.D. 50 refers correctly to the contents of the single chapters. According to MARTZLOFF (1988: 118ff), finally, certain parts of the work go back to early Han or even further, but precisely chapter VIII (and IV, which does not concern us here) contains no socio-cultural or metrological chronological cues suggesting an early date. Nor does its contents seem to be represented on the arithmetical bamboo strips found in a tomb from the second century B.C. (according to the preliminary information given in I & DU' 1987: 57). Since precisely chapter VIII is, as formulated by MARTZLOFF (1988: 124), »de loin, le plus original de tous«, it is probably safe to date it to the first century C.E.

[47] The different guises should not astonish us. In general, the editors of the *Nine Chapters* seem to have used their phantasy most creatively in order to vary the clothing of problems. But the Chinese collection shares many problem types apart from those mentioned here with the recreational mathematics characteristic of the other parts of the »Silk Road area«. A striking coincidence in *mathematical structure* alone should therefore be sufficient evidence that a particular Chinese problem is connected to one known from the remaining area.

should have had direct access to the Chinese textbook for future mandarins.

Diophantos' method, as we saw, was close to that of earlier Greco-Egyptian algebra, employing the same symbol for the unknown number and using it in the same way. This method, in contradistinction to that of the *Nine Chapters*, thus seems to have been in use among practitioners in at least part of the Classical world. It may also have accompanied the cluster of »Silk Road problems« further down through time. Leonardo, indeed, during his discussion of »give and take« problems, presents what turns out to be exactly the procedure used in *Arithmetica* I,xv (Diophantos' version of the »give and take«) under the name *regula recta*, telling it to be most commendable and in use among the Arabs[48]. Further on in the *Liber abaci*, the name and the method turns up repeatedly, showing the term to cover rhetorical algebra of the first degree and in one variable based on the name *res* for the unknown, corresponding to the Arabic *šay'* and later Italian / German *cosa / coss*, and calculating »de principio ad finem questionis«[49].

The presence of sub-scientific »Silk Road« material in the Classical world is confirmed by two slightly post-Classical sources. One of them is the above-mentioned collection of arithmetical epigrams in Book XIV of the *Anthologia graeca*[50]. The collection was presumably put together by Metrodoros around A.D. 500, but the single epigrams are of earlier and varied origin. More will have to be said about the collection below, but in the present connection it should be observed that two of the epigrams (N^{os} 145 and 146) are of the »give and take« type, while N^{os} 7, 130, 131, 132, 133 and 135 deal with the »filling of a container« from a number of sprouts with different capacity – a type which is also testified as N^{o} VI,26 of the *Nine Chapters*[51].

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[48] *Liber abaci*, ed. BONCOMPAGNI 1857: 191.

[49] *Liber abaci*, ed. BONCOMPAGNI 1857: 203. Here, the *regula recta* is opposed to the *regula versa*, the backward computation from result to unknown values. The »direct« computation, which begins at the beginning where essential nubers are still unknown, evidently presupposes that these be represented by some all-purpose name or symbol.

[50] Ed., transl. PATON 1979.

[51] Ed., transl. VOGEL 1968: 68. A mathematically analogous type treats of combined work performance. This is found as Anthologia graeca XIV,136 and in the Nine Chapters as N^{os} VI,20-25. It may be of interest that the solution in the case of two

The other post-Classical source of interest is the Carolingian collection *Propositiones ad acuendos iuvenes*[52], maybe put together by Alcuin (in any case connected to the Carolingian educational effort) but similarly composed from older (rather disparate) material circulating in the north-western provinces of the Roman Empire and adopted into Monastic recreational lore in late Antiquity. In any case, an Ancient origin can safely be ascribed to those problem types whose geographical distribution connects them to the transcontinental trading network, the characteristic of the early Frankish Middle Ages being precisely the extreme attenuation of international commercial relations – with ups and downs, it is true, but on an extremely modest level compared with the situation which had prevailed during the Principate.

One of these types is represented by another »give and take« problem (N° 16). Two others are the »hundred fowls« (N^os 5, 32, 33, 34, 38, 39, 47) and the »pursuit«. In the first type[53], a number (typically one hundred) of animals or objects (typically fowls) are bought at different prices per piece for different categories but totalling the same number of monetary units (N° 39, dealing with »animals bought in the Orient«, has camels at a price of 5 *solidi*, asses at one per *solidus*, and 20 sheep per *solidus*). In the second type[54], a pursuer and a pursued person or animal move at different paces, and the moment of catching up is asked for. At the simplest level

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workers doing a job together (or two sources filling the vessel) can be conveniently expressed by ways of the harmonic mean.

[52] Ed. FOLKERTS 1978.

[53] See TROPFKE / VOGEL 1980: 613ff.

Just before adding the final touch to the manuscript I was informed by JEAN CHRISTIANIDIS (1991: 8) that a problem of type »a hundred fowls« is indeed found already in a second century Greek papyrus. The same manuscript (p. 9) contains a highly attractive reinterpretation (supported by the third century grammarian Athenaeos) of Plato' *Laws* 819B4−6 (ed., transl. BURY 1967: II, 104f), according to which the adjustment of »the same totals« of apples or chaplets to »larger and smaller groups« refers to the »problem of remainders« familiar from a third century Chinese source and from later Indian and Arabic authors (TROPFKE / VOGEL 1980: 636ff): To find a number which leaves given remainders when divided by a sequence of given divisors.

[54] TROPFKE / VOGEL 1980: 588ff.

both speeds are constant[55]; this is the case in the *Propositiones* (N° 26, a hound pursuing a hare), but elsewhere arithmetically increasing and decreasing speeds occur. Both types are also testified in early Chinese mathematics, the »hundred fowls« in a treatise by the late fifth century author Zhang Qiujian[56], and the pursuit in several versions in the *Nine Chapters* (III,12 is the simple version corresponding to *Propositiones* N° 26, while III,14 is a more complex problem dealing with hound and hare).

A curiosity of some consequence in the *Propositiones* is N° 8. It borrows the dress of the »filling of a vessel« (actually, the vessel is not filled but emptied through three outlets of unequal capacity), showing thus familiarity with that tradition. But the mathematics of the usual filling problem being apparently too difficult, the mathematical substance is changed into something simpler. More significant, however, is N° 13. This is one of the many trigesimal doubling problems spoken of in the beginning of section III, and which should now be presented in more detail.

The oldest appearance of the problem is in a cuneiform tablet from Old Babylonian Mari[57] and runs as follows:

> To one grain, one grain has been added:
> Two grains on the first day;
> Four grains on the second day;
> .
> .
> .

going on until 30 days, but expressing the larger amounts not in numbers but in metrological units (when used as a weight unit, a »grain« is $^{1}/_{180}$ of a *šeqel*, itself some 8 g).

The following occurrence is a tabulation found in a Greco-Egyptian papyrus[58] (probably to be dated to the Principate but perhaps as late as the fourth century). It starts at 5 drachmas, contains again 30 steps (nothing

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[55] In this simple case, the reverted problem (»meeting«) comes mathematically close to the filling of a vessel from two sources.

[56] See TROPFKE / VOGEL 1980: 613f.

[57] Ed. SOUBEYRAN 1984: 30; my translation.

[58] P. Ifao 88, ed. BOYAVAL 1971.

is said about days) and makes use of the copper talent (=6000 drachmas) and a still larger unit of 13200 talents when reaching sufficiently large numbers.

Next in time follows *Propositiones* N° 13[59]:

> A certain king ordered his minister to gather an army from 30 domains in such a way that from each he should levy as many men as he brought to there. But he came alone to the first domain, and to the second with another man; now three came [with him] to the third. Let the one who is able to say how many men were gathered from the 30 domains.

Then, finally, we have al-Uqlīdisī's observation, made in tenth century Damascus, that »many people« ask for 30 (or 64) doublings of unity, and the indisputable presence of the problem everywhere in the »Silk Road« area. As already stated above, the possibility of independence can be safely disregarded. At least one of the problems belonging to the Medieval »Silk Road« cluster can thus be demonstrated to have a Babylonian (or even older) origin; and to have been widespread within the Roman Empire (from Egypt to the northwestern corner). Others, well known from Medieval and Ancient Chinese sources, have left their traces in Diophantos *Arithmetica*, the *Anthologia graeca*, and elsewhere in the *Propositiones*.

We may conclude that a whole fund of sub-scientific mathematics, connected to the transcontinental trade routes and including a superstratum of »recreational«, non-utilitarian problems, was diffused throughout Greco-Roman society though at the »culturally subliminal« level. It might be worth asking whether it is reflected in other ways. One possibility was already hinted at: The relation between the concept of the harmonic mean and the problems of combined performances or the »filling of a vessel«. Another example might be Zeno's paradox of Achilles and the tortoise; the point of this might be even sharper than usually assumed if it does not refer to common sense understanding only but also to the ways of vulgar computation. More close at hand than both possibilities is, however, the possibility (equally touched on above) that the algebraic αριθμός-technique used Diophantos (and by that tradition which he hints at in the

[59] Ed. FOLKERTS 1978: 51. My translation.

introduction, including Papyrus Michigan 620) was borrowed from the same sub-scientific tradition, which also transmitted it as Leonardo’s *regula recta*[60]♦.

V. Composite fractions

Other diffusion networks span smaller regions. One is connected to a particular idiom for fractions, best known from Medieval Arabic sources and from Leonardo Fibonacci[61]. Its basis is the »composite fraction«, »$^1/_p$ of $^1/_q$« instead of »$^1/_{(p\cdot q)}$«, and its most highly developed form the »ascending continued fraction«[62], »$^p/_q$, and $^r/_s$ of $^1/_q$, and $^t/_u$ of $^1/_s$ of $^1/_q$« (or going on to even more members) – in numerical examples »$^1/_3$ of $^1/_5$« and »$^2/_5$, and $^4/_5$ of $^1/_5$«. Though rare, both varieties also do turn up in a number of Old Babylonian sources – either as a final recourse when other notations fail or in problems of riddle character, which fits an existence as a popular

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[60] It should be emphasized that this is a suggestion and no firmly proved fact. The regula recta is, after all, only brought in as a secondary method, not as the technique going »naturally« with the problem, and at the moment where Leonardo made himself a disciple of the Arabs al-Khwārizmī’s Algebra (which uses the technique amply) had already circulated for almost 400 years.

On the other hand, the relation between the »thing« terminology (used, e.g., exclusively in the treatment of inheritance problems) and the »treasure and root« terminology used when second degree problems are solved suggest that these terminologies and techniques are of different origin. Of particular interest are problems of the type »I have divided ten into two parts, and multiplying one of these by the other, the result was twenty-one«. In the first step, such division problems (which have a definitely Diophantine ring, cf. Arithmetica I,xxvii) are first expressed in »thing« terminology (one number is the »thing«, the other is »ten minus a thing«) and next translated into »treasures and roots«.

Detailed documentation of these suggestions would lead too far; but as far as the present issues are concerned, ROSEN’s translation (1831) can be safely used, even though ROZENFELD’s Russian (1983) or Gerard of Cremona’s Latin (in HUGHES 1986) are to be preferred (Gerard, however, omits the legacy part). Cf. also HØYRUP 1990b.

[61] For more complete documentation and discussion I refer to HØYRUP 1988.

[62] It should be emphasized that this concept has *nothing* to do with ordinary continued fractions except for the graphic similarity between the two when both are written with fraction lines. Ordinary continued fractions are a way to write down the outcome of an *anthyphairesis*-procedure (use of the »Euclidean algorithm«); »ascending continued fractions« are a generalization of the principle of measurement by a system of decreasing units.

or sub-scientific usage known to the scribes but not accepted as fully legitimate by them. In Medieval Arabic, composite fractions are evident as normal language – »one third of one fifth« *was* simply the current name for $^1/_{15}$. Ascending continued fractions went together with the »finger reckoning tradition« preferred by merchants. Once more we encounter a popular usage and connections to a sub-scientific practice.

Even in Egypt, composite fractions turn up (though only rudimentary ascending continued fractions). Again, it happens when popular usage is portrayed (a herdsman speaking to an official defines his due as »$^2/_3$ of $^1/_3$« of the cattle entrusted to his care) or in the riddle »go down I [*viz.*, a jug of unknown capacity] times 3 into the *ḥeḳat*-measure, $^1/_3$ of me is added to me, $^1/_3$ of $^1/_3$ of me is added to me, $^1/_9$ of me is added to me; return I, filled am I. Then what says it?«[63] (i.e., $3+^1/_3+^1/_3\cdot^1/_3+^1/_9$ times an unknown quantity equals 1 *ḥeḳat*).

Once again, we seem to be confronted with a popular usage, normally avoided by the scribes when they had developed that sophisticated unit fraction system which was eventually borrowed by the Greeks. Evidently, the composite fractions and the additive unit fraction system differ fundamentally; but some evidence exists that the latter system developed from the same set of simple unit fractions ($^1/_2$, $^1/_3$, $^1/_4$, $^1/_5$ and $^1/_6$ – and, notwithstanding *our* conceptions of system, $^2/_3$) which was extended in more popular usage through multiplicative composition.

I have not come across the system in texts from the Classical epoch, but it turns up in *Anthologia graeca* XIV as well as in the *Propositiones* – strangely enough in two different versions.

Strictly speaking, it is not the ordinary system which is found in the *Anthologia* but a curious travesty: »Twice two-third« (N° 6); »One-eighth and the twelfth part of one-tenth« (N° 121); »The fifth part of seven-elevenths« (N° 128); »Twice two-fifths« (N° 129); »A fifth of the fifth part« (N° 137); »Four times three-fifths« (N° 139); »Twice two-sixths and twice one-seventh« (N° 140); »Six times two-sevenths« (N° 141); »A fifth part of three-eighths« (N° 142); and »Twice two-thirds« (N° 143). Everywhere else, fractions are expressed in the usual Greek (and Egyptian) manner.

The choice of one or the other usage has nothing to do with the

[63] Rhind Mathematical Papyrus, N° 37, in CHACE's literal translation (1929, Plate 59). The herdsman is put on the stage in N° 67.

mathematical substance of the problems (most of which are anyway of the same type, reducible in symbolic form to an equation $x \cdot (1-p)=A$, where p is a sum of fractional expressions). Nor is it, however, random: it is geared to the clothing of the problem. Composite fractional expressions turn up in all problems dealing with the Mediterranean extensions of the Silk Road (N^{os} 121 and 129), with the legal partition of heritages (N^{os} 128 and 143), and with the hours of the day (N^{os} 6, 139, 140, 141, and 142; N^{o} 141 is connected to astrology). A final instance is found in N^{o} 137, dealing with a catastrophic banquet apparently meant to be held in Hellenistic Syria. Problems which refer to Greek mythology or history make use of Greek / Egyptian fractions. The same applies to problems dealing with apples or walnuts stolen by girl friends, with the filling of jars or cisterns from several sources, with spinners', brickmakers' or gold- or silversmiths' production, with wills, and with the ages of life.

The most plausible explanation of this striking distribution is that a number of recreational problems belonging to (at least) two different professional contexts (providing the guises of the problems) have been brought together in the anthology, each conserving its own distinctive idiom for fractions: on one hand the traditional Greek idiom based on unit fractions (and occasional rudimentary general fractions); on the other the usage of the trading community and of juridical calculators, and of astrologers and makers of celestial dials, which is different. The association between astrology and »Chaldaeans« as well as the Syrian banquet and the use of composite fractional expressions in juridical calculations from Seleucid Uruk suggest that the origin this usage (and hence the source for the corresponding technologies of time-measurement etc.) should be looked for in the Semitic-speaking Near East. Here, as we have observed, the system of composite fractions had indeed been in use at least since the early second millennium B.C.

Of course, the composite expressions found in the *Anthologia graeca* can not be expected to have been those in practical use among traders etc. It is not conceivable that »two-thirds« should be expressed as »twice two-sixths« for any everyday purpose. But even the »Greek« group of epigrams contains similar deviations from computational »real life«. The »double« and »triple« seventh of N^{os} 116 and 119, of course, are fairly regular, as are the »two fifths« of N^{o} 132; but the »double sixth« and the »two quarters« of N^{os} 117 and 119 are certainly not (N^{os} 116, 117 and 119 deal with division of apples, and N$^{\rm o}$ 132 with the filling of a cistern). Most likely, the irregular expressions are to be explained from the recreational character of the problems: by being queer, they make the riddles more funny and more obscure.

The composite fractional expressions seem to have remained strangers in the Classical world, and to have been unable to spread from those specific groups of practitioners who brought them or adopted them along with other techniques. Admittedly, composite fractions are also found in the *Propositiones*, – but in a way which suggests an ultimate root in Egypt (the way from Egypt to Charlemagne’s Aachen may of course have been highly tortuous). They are found in N$^{\rm os}$ 2, 3, 4 and 40, which all belong to the same type. We may quote N$^{\rm o}$ 40 as an example[64]:

> From a mountain, a man saw sheep grazing, and said: If only I had as many and as many once more, and half of the half, and further the half of that half, and then I would enter my house together with them as one of hundred. Let the one who is able to find out how many sheep he saw grazing there.

Thus, the unknown number taken twice, with its $^1/_2\cdot{}^1/_2$ and its $^1/_2\cdot{}^1/_2\cdot{}^1/_2$ is 99. The fraction is the same sort of rudimentary ascending continued fraction as found in the Egyptian *hekat*-problem – and the mathematical structure of the problem is also strikingly similar[65]. The composite fractions found in N$^{\rm os}$ 2, 3 and 4 are $^1/_2\cdot{}^1/_2+{}^1/_2\cdot{}^1/_2\cdot{}^1/_2$, $^1/_3+{}^1/_2\cdot{}^1/_3$ and $^1/_2\cdot{}^1/_2$, respectively, i.e., two of the same type and one reduced.

The quintuple occurrence of the same problem type (N$^{\rm os}$ 45 is similar but only contains the fraction $^1/_3$) shows that it must have been quite

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[64] Ed. FOLKERTS 1978: 68; my translation.

[65] On the abstract mathematical level, of course, both belong the class of so-called `ḥ`-problems, $a\cdot x=b$ (`ḥ` is Egyptian for »heap« or, more abstractly, »quantity«; the term refers to problems in the Rhind Mathematical Papyrus dealing with such indefinite »heaps«). This class, however, is too wide-spread and too unspecific to be evidence of anything. Conclusions only become possible if the precise structure of a is noticed. The regular Egyptian `ḥ`-problems have $a=1+{}^1/_p+{}^1/_q...$; the apple problems of the *Anthologia graeca* have $a=1-{}^1/_p-{}^1/_q...$; problems with $a=n+r$, where r stands for a »rudimentary« ascending continued fraction are rare, in fact known to me only from the Rhind Papyrus and from the *Propositiones*.

popular. It remained so in later Medieval problems collections[66], retaining even the *numerical value* of the characteristic series of fractions; but instead of speaking of »$^1/_2$ and $^1/_2$ of $^1/_2$«, the »Columbia Algorism« speaks simply of »$^1/_2$ and $^1/_4$«[67]. Once again, the peculiar technique of composite fractions proved able to survive for a while when attached to a specific and isolated tradition – which must have been the situation in the Classical world; but when the tradition in question left the Hamito-Semitic language area and was absorbed into a broader current, the composite fractions were replaced by more familiar expressions. The occurrence of composite fractions in the West is thus a reliable index fossil, demonstrating the survival of an autonomous sub-scientific tradition (*different* traditions, indeed, if we look at the *Anthologia* and the *Propositiones*[68].

VI. »Surveyors' algebra« and »calculators' algebra«

After the sequence of propositions apparently inspired by »recreational« first degree problems in several variables in Diophantos' *Arithmetica* I comes a sequence dealing with problems of the second degree: To find two numbers with given sum and product (xxvii); to find two numbers, when their sum and the sum of their squares are given (xviii); to find two numbers, when their sum and the difference between their squares are given (xxix); to find two numbers with given difference and product (xxx); and to find two numbers with a given ratio, when the sum of or difference between their squares has a given ratio to their sum or their difference (xxxi-xxxiv), or when the square of the smaller number has a given ratio to the smaller or greater number or to their sum or difference (xxxv-xxxviii).

In his introduction, Diophantos promises to teach the solution of mixed

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[66] TROPFKE / VOGEL 1980: 574f lists its occurrence in the tenth-century Iranian al-Ṭabarī, the twelfth-century Spanish-Hebrew ibn Ezra, and in a 14th-century Italian algorism.

[67] Ed. VOGEL 1977: 109.

[68] A third instance could be pointed out. As mentioned above, Leonardo Fibonacci introduced the ascending continued fractions in his Liber abaci, together with an ingenious notation borrowed from the Maghreb school of mathematics. From the Liber abaci they went into the Italian abacus school, in itself a sub-scientific institution; there they survived until the sixteenth century (Clavius still discusses them), ultimately to disappear when this sub-scientific tradition dissolved in the late Renaissance (see VOGEL 1982).

second-degree equation with one unknown[69] (a promise which he does not keep in the conserved parts of his text); at various places in Book VI, furthermore, he refers to the solvability conditions for non-normalized second-degree equations in one variable, and at others he states actual solutions of such equations without explanation[70]. Apart from that, however, non-trivial (i.e., mixed) numerical second-degree problems only turn up in utterly few Greco-Roman sources. One place is the quasi-Heronian compilation *Geometrica*, where the same problem turns up twice, in **21**,9 and again in **24**,46[71]: To find the diameter of a circle when the sum of the diameter, the perimeter and the area is 212. The solution follows from a numerical algorithm given without comments, but corresponding to the way we would treat the problem $(11d)^2+2\cdot 29\cdot(11d)=32648$ (which agrees with the original statement if $\pi={}^{22}/_{7}$).

Two other sources both deal with right triangles. One is the anonymous *Liber podismi*[72]. This opuscule is part of the *Corpus agrimensorum*, which was collected in the mid-fifth century C.E. from older material. One of the problems dealt with refers to a right triangle, whose hypotenuse and area are given. Algebraically, the problem can be expressed as $x+y=A$, $x^2+y^2=B$; but the solution seems to build on a simple piece of geometrical insight, which follows from this diagram:

Figure 1

[69] Ed., transl. TANNERY 1893: I, 14f. Possibly, however, Diophantos had something more complex in mind, as suggested by SESIANO (1982: 78).

[70] Ed., transl. TANNERY 1893: I, 392-449. References to solvability conditions are found, e.g., in VI,vi and VI,xxii, actual solutions, e.g., in VI,vi and VI,vii.

[71] Ed., transl. HEIBERG 1912: 380f and 444-447.

[72] Ed. BUBNOV 1899: 510-516, cf. p. 399 and FOLKERTS 1970: 95-98 on manuscripts and authorship. The problem mentioned is in BUBNOV 1899: 511f.

If H is the square on the hypotenuse, and A is the area, then $H+4A=(x+y)^2$, and $H-4A=(x-y)^2$.

The other text dealing with right triangles is the Greco-Egyptian Papyrus Genève 259, which contains three problems and should probably be dated to the second century C.E.[73]. We may denote the hypotenuse c and the other sides a and b. The first problem (a=3, c=5) is trivial once the Pythagorean theorem (which also follows from the above diagram) is known, and the third ($a+b$=14, c=10) is too damaged to allow any certain reconstruction[74]. But the solution of the second ($a+c$=8, b=4) appears to make use of the rule that $b^2=c^2-a^2=(c+a)\cdot(c-a)$, which can be claimed to be algebraic in nature, but which (given the Pythagorean theorem) can be easily ascertained on a diagram similar to the above.

It is not conceivable that these isolated Latin and Greek geometrical computations should have popped up from nowhere – their way of obtaining the solutions from unexplained sequences of numbers demonstrates that well-known procedures were used. Together the two sources thus establish the existence of yet another concealed mathematical undercurrent, somehow connected, it appears, with practical geometrical computation. In this context, they seem to have belonged to the non-utilitarian superstratum – a practical geometer will hardly ever know the sum of the hypotenuse and another side of a right triangle before he knows them separately, nor need to construct one from such data. Isolated and laconic as the texts in question are they tell us little more – in particular not whether the methods were really founded on insights of a geometrical nature or on an αριθμός-algebra *à la* Diophantos.

Once more, sources from earlier and later epochs will be of help, showing us the river before it goes underground and after it reappears. At the same occasion, they will inform us about some of Diophantos' sources and, so it appears, about other aspects of the history of Greek mathematics.

The central elements in the argument will be the Old Babylonian second

[73] Published, translated and discussed by RUDHARDT (1978). Further discussion in SESIANO 1986.

[74] RUDHARDT suggests the first step to be a squaring of the hypotenuse. But all that is sure is a ρ, which might just as well (and no less reasonably, cf. below, note 83) be the first »digit« of 196, the square on $a+b$. SESIANO's complete reconstruction is pure conjecture.

degree »algebra« and an Arabic text written by an unidentified Abū Bakr and known from a Latin translation *Liber mensurationum* due to Gerard of Cremona.

Since NEUGEBAUER's and THUREAU-DANGIN deciphered and interpreted the Babylonian mathematical texts in the 1930es, it has been a prevailing belief that the whole class of texts dealing with squares and their sides and with rectangular lengths, widths and areas was nothing but algebra in geometrical disguise, and it has been taken for implicitly granted that »algebra« would treat of *numbers*, and would, if it did not possess the modern (Cartesian) symbolism, do so by means of »rhetorical« techniques in the vein of Diophantos' αριθμός-algebra and the Arabic *šay'*- / thing-representation.

A detailed comparative investigation of the »algebraic« texts shows this conclusion to be precipitate and even erroneous[75]. Their rectangles and squares are not metaphors for products and second powers of numbers but real geometrical figures (abstract »fields«, in fact). The procedures which are described, furthermore, are not numerical algorithms but reports of geometrical cut-and-paste procedures. As an example we may look at the simplest text of all[76]:

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[75] The argument for this is complex, involving a structural investigation of the total terminology and a close comparative reading of many texts. Part of the outcome of this investigation is that the Old Babylonian scribal mathematicians distinguished two different »additive operations« (i.e., operations which when read as operations with abstract numbers are both additions), two different subtractive operations, and no less than four different »multiplications«. Nothing of this makes sense in a numerical interpretation, where there *is* only one addition, one subtraction, and one multiplication. But if one »multiplicative« operation consists in constructing a rectangle, another one in repeating a geometrical figure concretely (e.g., by joining it to a mirror image), a third in calculating a concrete magnitude through an argument of proportionality, and a fourth in making repeated additions of a number, then the operations *are* really different, and it makes sense to label them differently.

The details of the investigation are presented in HØYRUP 1990; a summary exposition will be found in HØYRUP 1989.

[76] The first problem from the tablet BM 13 901 (ed. NEUGEBAUER 1935: III, 1-5). The translation is my own, and is extremely literal, except for the numbers (the particularities of the Babylonian numerical notation are irrelevant in the present connection). The tablet contains a long sequence of problems dealing with one or more squares, and we shall have to return to it repeatedly.

[1] The surface and my confrontation I have accumulated: $^{3}/_{4}$.
[2] 1 the projection you pose.
[3] The moiety of 1 you break, $^{1}/_{2}$ and $^{1}/_{2}$ you make span,
[4] $^{1}/_{4}$ to $^{3}/_{4}$ you append: 1 makes 1 equilateral.
[5] $^{1}/_{2}$ which you have made span, from the body of 1 you tear out:
[6] $^{1}/_{2}$ is the confrontation.

This cries for explanation. The »confrontation« (*mitḫartum*) is a configuration characterized by the confrontation of equal sides, i.e., a geometrical square. But since the Babylonians understood the magnitude of a square as characterized by its side as distinctive parameter, the »confrontation« is also identified with the numerical value of the side. This is less strange than we may find at first. To us (and mostly to the Greeks), a square (which is after all a complex geometrical configuration with sides, angles, diagonals, area, circumscribable circle etc.) *has* a side of two feet and *is* four square feet; to the Babylonians, on the other hand, it *had* a surface of 4 square feet and *was* two feet. We shall return below to a specific geometrical Greek term (the δύναμις) which reflects the same understanding.

In **[1]** we are thus told that a square has a sum of the numbers measuring the area and the side equal to $^{3}/_{4}$ – »accumulation« (*kamārum*) is the real addition among the two »additions«, and it allows the addition of numbers without regard for their significance. The rest of the text is best explained on a diagram:

Figure 2

In **[2]**, the square (whose side we may for brevity designate with a Cartesian x) is provided with a »projection« (*waṣītum*) of 1. As we see, this corresponds to appending a rectangle of length 1 and width x, i.e., of area $x \cdot 1 = x$. The area of the total figure will then be $x^2 + x$, which is known to be $^3/_4$.

Next **[3]** this »projection« is »broken« (*ḫepûm*) into »moieties« (*bāmtum*). »Moieties« (literally rather »rib-sides«) are »natural« or »customary« halves, as the radius of a circle is the natural half of a diameter. »Breaking« is bisection into »moieties« (the two terms thus go together). The two moieties (with appurtenant sections of the rectangle) are »made span« (*šutākulum*), i.e., they are used to form a rectangle (actually a square), whose area is seen **[4]** to be $^1/_2 \cdot ^1/_2 = ^1/_4$. When this is »appended« (*wasābum*) to the area ($^3/_4$) of the transformed figure, the outcome is a larger square with area $^3/_4 + ^1/_4 = 1$. This area »makes 1 equilateral« (íb-si$_8$), i.e., if it is formed as a square it causes 1 to be the side of this square. Finally **[5]**, that part of the broken rectangle which was moved and »made span« is »torn out« (*nasāḫum*) from the »body« (*libbum*, literally »heart« or »bowels«) of the side of this larger square (meaning from the concrete, bodily entity, not from a measuring number), **[6]** leaving the original unknown »confrontation«, which thus equals $^1/_2$.

The correctness of the procedure is intuitively obvious, even though it is »naive«, as opposed to the »critical« approach which characterizes Greek mathematics (Euclid, in the very similar proof of *Elements* II,6, does not loosely move a rectangle, but constructs another one, proving it to have the same area, etc.[77]). The method is *analytic*, i.e., that which is unknown is taken to be known and moved around until something really known eventually drops out – as it happens when we represent an unknown number by x and write down what we know about its relations. It is, moreover, homomorphic with the analytical procedure which *we* would apply: $x^2+x=^3/_4 \Rightarrow x^2+2\cdot(^1/_2\cdot x)+(^1/_2)^2=^3/_4+^1/_4=1 \Rightarrow (x+^1/_2)^2=1 \Rightarrow x+^1/_2=\sqrt{1}=1 \Rightarrow x=1-^1/_2=^1/_2$.

In the scribal school, a highly systematic teaching was built up around these techniques. The aim was not to create *theory*, but it was still non-utilitarian; just as the mastery of written and spoken Sumerian, proficiency in second-degree »algebra«, so it seems, was one of the ways in which a scribe could display professional virtuosity. But certain indications exist[78]

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[77] Ed., transl. HEIBERG 1883: I, 132ff; cf. HEATH (ed., transl.) 1926: I, 385f.

[78] See HØYRUP 1990a: 79f.

that the techniques did not originate inside the Babylonian school but were taken over from a non-scholarly sub-scientific tradition (carried, we may surmise, by surveyors and other practical geometers), where it served in more genuinely recreational problems[79].

As it was argued concerning accountants and merchants, the mathematical problems used in everyday practice by a surveyor will soon become trivial. Everybody within the craft will be familiar with the determination of a rectangular area from length and width, and will be able to add up partial areas. In order to demonstrate professional dexterity beyond the ordinary level you should be able to answer more specious questions, which, in agreement with the familiar psychology of recreational mathematics, should at the same time contain something striking. A first question of this type would be precisely to ask for the side when you known the sum of the area and the side of a square. But while the next question occurring naturally to a school teacher is then the sum of the area and another multiple of the side, and next the difference, and a multiple of the area together with a multiple of a side, the obvious next *funny* question concerns the sum of the area and *all four sides*.

The tablet containing as its first problem the »area plus side« exhibits both features. It proceeds systematically, exactly as a school text could be expected to do. Towards the end comes, however, precisely the question of area and four sides; the formulation, however, is unorthodox, and the procedure makes use of a special trick which only works in this case. The function is clearly that of entertainment during the »last lesson before Christmas«, and the language suggests the square field in question to be imagined as less abstract than the others. Everything fits a problem borrowed from a living, non-scholastic tradition.

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[79] In this context, the quadratic complement (the essential trick, in fact, in the solution of mixed second-degree problems) will have played a role similar to that of the intermediate stop in the camel problem from the *Propositiones*. The trick seems to have carried the name »the Akkadian method«, suggesting that it originated among Akkadian practitioners, not among the Sumerian scribes of the third millennium B.C. At the emergence of Akkadian scribe-hood in the Old Babylonian era, it will have followed the language into the school curriculum, making thus second-degree »field« problems the distinctive characteristic of Old Babylonian (as opposed to Sumerian) scribal mathematics.

This tradition proved astonishingly hard to kill. The *Liber mensurationum*[80] mentioned above, a work whose Arabic original was probably written around or shortly after A.D. 800, still appears to remember it. The evidence for this is multifarious.

Firstly, there is what might be called the »rhetorical structure« of the problem texts. The Old Babylonian text quoted above exhibited some characteristic features, which when more (and longer) texts are included amount to a system:

The text begins with, or presupposes, the phrase »If somebody / the teacher has said to you«. Then follows the statement of the problem, which is held in the first person singular of the preterit tense, *with one exception:* if the length of a rectangle exceeds the width by a certain amount, this is stated in the third person singular of the present tense as a neutral fact, not as something which the speaker has caused it to do. Then comes a phrase (implicit above) »you, by your method«, and then a description of the procedure, formulated in the present tense, second person singular, or in the imperative.

Occasionally, a certain step in the procedure is justified by a quotation from the statement. Such quotations are literal (grammatical forms included), and indicated by the phrase »because he has said«. At other points, an intermediate result is to be remembered, not taken down. This number is then followed by the phrase »which your head shall retain«.

At first sight, the corresponding structure of the *Liber mensurationum* is more complex. For one thing, the second part of this treatise deals with *real* mensuration of Heronian character, and thus does not concern us here. But apart from that, the first part combines two traditions. After the statement and the description of the procedure to be used for the solution of each problem comes in most cases the observation that »there is another method according to *aliabra*«, which is then described. The solution »according to *aliabra*« turns out to make use of *al-jabr* (»treasure-root-algebra«) as presented by al-Khwārizmī (but not exactly in his formulation). If we disregard this alternative method, however, the rhetorics of Abū Bakr's text follows the Old Babylonian scheme in every particular, with the sole exceptions that »your method« has become »the method«, and

[80] Ed. BUSARD 1968. The text is analyzed in HØYRUP 1986, and again in HØYRUP 1990b, to which publications I refer for the sake of documentation.

that »your head« has changed into »memory« in the Latin version.

As the basic fund of Old Babylonian second-degree algebra, Abū Bakr's problems deal with squares and rectangles (rhombs are treated too, but in fact trivially reduced to the rectangles in which they are inscribed). Apart from the determination of a diagonal from the side(s) or vice versa, and a few similar issues, the questions have no relevance for practical mensuration – they belong to the same family as the Old Babylonian »square area plus side« problem quoted above. That a number of simple problem types are shared (e.g., in symbolic interpretation, $x^2 \pm x = A$; and $x+y=A$, $x \cdot y = B$) is then in itself not astonishing: after all, the number of simple problem types concerning squares and rectangles is quite restricted. But more striking coincidences are found, involving certain very idiosyncratic Old Babylonian problems together with their no less idiosyncratic methods (amounting in modern language to a »change of variable«). Even the distinctions between two different additive and two different subtractive operations is found, together with traces of the distinction between different multiplications.

The text used by Gerard for his very literal translation must have been corrupt in several respects, as demonstrated by the presence of repeated and permuted problems. The most serious flaw is the absence of a number of diagrams to which the text refers. None the less, the text as it stands may make us confident that the basic method, the one that is used in the first solution of each problem, was precisely that »naive« cut-and-paste geometry which the Babylonians handled with such skill.

All in all, there can be no reasonable doubt that Abū Bakr had access to a tradition going back to the Old Babylonian era and used it as his fundament for the first part of his treatise (while demonstrating that the same solutions could be found by means of *al-jabr*). On the other hand, important Old Babylonian problem types are absent from his collection, most notably all mixed problems necessitating the use of operations of proportionality – in particular problems of the types $a \cdot x^2 + b \cdot x = c$ (BM 13 901 N° 3) and $x^2+y^2=A$, $y=p \cdot x+q$ (BM 13 901 N^os^ 9, 10, 11, 13, and 14); in other words, problems which cannot be solved by cut-and-paste geometry alone but involve changes of scale or complex coefficient accounting. At the same time, problems involving the sides of squares or rectangles will mostly involve *one* side, *one* length and *one* width, or *all four* sides; this is quite different from the style of the Babylonian school tablets, but agrees (as observed above in connection with a rare Babylonian specimen) with that predilection of genuine »recreational« traditions for striking formulations which was referred to above. We may hence conclude, *either* that the tradition which Abū Bakr used as his fundament did not derive directly from the Old Babylonian scribal tradition but from an even earlier sub-scientific source tradition from which even the Old Babylonian school had borrowed; *or* that the scribal mathematical tradition was fitted to the non-scholastic needs of that sub-scientific surveyors' environment which appears to have carried the tradition onwards after the collapse of the Old Babylonian school system.

Scribal-scholarly second-degree »algebra« turns up again in a few Seleucid texts, in a way which makes manifest a passage through a non-scholarly environment[81]. These Seleucid texts appear to represent in themselves a dead alley, but they derive from a stage of the tradition between what we know from the Old Babylonian era and Abū Bakr. One of them, in particular[82], exhibits a strong interest in the diagonal of the rectangle and in the right triangle, embracing in fact all the three problem types of the Genève papyrus though formulated as questions concerned with rectangles with diagonal and not with right triangles. Some very particular problems from this Seleucid tablet turn up once more in the *Liber mensurationum* – and the problem corresponding to No 2 of the Genève papyrus, which is solved there in a way which differs from that used in the cuneiform text (N^{os} 3, 4 and 11), is solved by Abū Bakr (N^{o} 30) precisely as in the papyrus[83].

It can thus be taken for granted that both the Genève papyrus and the *Liber podismi* reflect the presence in the Greco-Roman world of that

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[81] A characteristic of the Babylonian scholarly environment is the use of Sumerian terms for spoken Babylonian. But some of the of Sumerograms in the Seleucid texts turn out to be results of a recent retranslation: e.g., a term which in Old Babylonian texts had meant »repetition« (one of the four »multiplications« has suddenly come to mean »addition«, which is, in fact, a possible extension of its general semantics but not of its established meaning as a mathematical *terminus technicus*.

[82] BM 34 4568, ed., transl. NEUGEBAUER 1935: III, 14-22.

[83] Abū Bakr's No 28, whose statement coincides with the damaged N^{o} 3 from the Genève papyrus, begins (like N^{o} 10 of the Seleucid tablet) by squaring the sum of the sides, and not by squaring the diagonal, as RUDHARDT and SESIANO conjecture for the papyrus problem. But as observed in note 74, the conserved papyrus text fits one beginning just as well as the other.

surveyor's tradition which connects Abū Bakr with Old Babylonia. Similar roots can be claimed for the quasi-Heronian circle problem. This very specious problem is, in fact, found in the Old Babylonian catalogue text BM 80 209[84]. It is important to observe, however, that this problem is not normalized, and thus calls for an operation of proportionality. It can only have survived in a less reduced descendant of the Old Babylonian tradition than the one Abū Bakr had access to.

Because of their closeness to the *Liber mensurationum* it can also be safely assumed that both the Genève papyrus and the *Liber podismi* base their method on »naive« geometrical understanding. Since »Hero« relates differently to the tradition, we are on less firm ground in his case.

Hero's way to deal with the normalization reminds of the Old Babylonian technique, which consists in multiplying with the coefficient to the second-degree term instead of eliminating it[85]. The same technique is presupposed by Diophantos when he states solvability conditions for second-degree equations in *Arithmetica* VI. In itself this proves little; yet Diophantos' way to state the solution to these equations without further ado suggests that he refers to well-known procedures, and the indubitable Babylonian inspiration behind »Hero« together with the terminology for powers shared by Hero and Diophantos[86] indicates that these procedures are of Babylonian origin. What then about the second-degree problems with two unknowns in *Arithmetica* I?

In Old Babylonian »algebra«, rectangles with known area and known sum of or difference between the sides abound. Translated into numbers and their product, this corresponds to *Arithmetica* I,xxvii and xxx. In both cases, Diophantos proceeds via the semi-sum and the semi-difference between the two unknown numbers (of which, in both cases, the one is known and the other taken to be the αριθμός). This agrees with what the

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[84] Ed. FRIBERG 1981: 61.

[85] Thus transforming (in the symbolic interpretation) $ax^2+bx=c$ into $X^2+bX=ca$, with $X=ax$.

[86] See the use of the term δυναμο-δύναμις in Hero, *Metrica* I,xvii (ed., transl. SCHÖNE 1903: 48f) and in Diophantos' introduction, ed., transl. TANNERY 1893: I, 4-7. (LIDDELL and SCOTT's *Greek-English Lexicon* mentions no other authors using the term). That he had his terminology for the powers of the unknown from established custom is actually what Diophantos himself tells (cf. HØYRUP 1990c, note 9).

Babylonians had done. In *Arithmetica* I,xxviii, Diophantos asks for two numbers, of which the sum and the sum of their squares are known ($x+y=20$, $x^2+y^2=208$ is the example given). The same problem occurs as N° 8 on that Old Babylonian tablet (BM 13 901) whose N° 1 was quoted above, although solved in a slightly different way.

In all three cases, a diorism (solvability condition) is stated, followed by the remark that this is πλασματικόν, which might mean that it can be verified on a diagram[87]. All three diorisms, indeed, follow easily from the πλάσμα (standardized diagram) which is shown above in Figure 1, and which is also familiar from Old Babylonian texts.

Arithmetica I,xxxix, where Diophantos asks for two numbers, of which the sum and the difference between their squares are known, has no known Old Babylonian parallel. Nor does it state anything about being πλασματικόν; but since no diorism is needed there is no pretext to state it. From internal evidence alone it is, all in all, difficult to claim that the cluster I.xxvii−xxx must by necessity be inspired from a tradition going back to the Babylonians. If we look at the totality of Book I, however, where the initial first-degree problems seem inspired by current recreational and similar mathematics, and where the trivially casuistic sequence I,xxxi−xxxviii could have been represented by one or two specimens without theoretical loss, it is a reasonable assumption that the whole book is inspired by existing sub-scientific traditions. Some problems may have been taken over directly without any other change than the removal of the concrete interpretation of numbers (this is obviously the case in xv, the »give and take« problem, and in xxiv, the »purchase of a horse«). Others may perhaps have been developed as analogues and generalisations by Diophantos himself and included for completeness' sake.

The triangular problems in *Liber podismi* and the Genève papyrus appeared to belong to a practical geometers' tradition, while the »purchase of a horse« (etc.) would rather go with traders, calculators and accountants. Diophantos, however, would use the same αριθμός in problems of all

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[87] This is VER EECKE's interpretation (1926: 38 n. 3). Because the distribution of the term in the Arabic books of Diophantos' *Arithmetica* agrees badly with an interpretation through Euclidean geometry, both editors of that text have rejected VER EECKE's proposal (see RASHED 1984: III, 133-138 and SESIANO 1982: 192f); but if »naive« geometry in the style of Figure 1 is meant, their objections are not compelling – see HØYRUP 1990, chapter X.3.

degrees; his δύναμις, the second power of the αριθμός, is told by Plato to have been used by calculators in this function already around 400 B.C.[88] Whatever their origins, the various traditions drawn upon for *Arithmetica* I will thus have been merged by practical calculators already in the early Classical period into *one* field. This will have been the source for Hero's second-degree equation and for both categories of second-degree problems in Diophantos; the simple surveyor-»algebra« will apparently have followed a separate way.

It has usually been assumed that Diophantos took his term δύμανις from geometry. Here, too, the term was used from early times, and it has been much discussed whether it meant »square« or »square root«/»side of square«. The puzzle is solved if one observes that all occurrences of the »geometers' δύναμις« fit the use of the Babylonian *mitḫartum* – a square identified *by* and hence *with* its side. A thorough discussion of this and of the relation between »calculators'« and »geometers' δύναμις« would be extensive[89]; the main outcome is that the calculators' concept seems to be primary, and to have served in a naive-geometrical »algebra« of Babylonian type and descent. It will then have been borrowed by geometers in the late fifth century B.C. and used when they launched the enterprise which eventually gave rise to *Elements* II (etc.). This so-called »geometrical algebra«, as it has been called, will not have been a »dressing in geometrical garment« of a Babylonian numerical algebra, as it has repeatedly been maintained since the discovery of Babylonian second-degree »algebra« (and has in recent decades been vehemently denied). It will rather have been a critical investigation of the foundations of the naive-geometrical procedures of the calculators, which was then worked up as a discipline of its own with its own systematics and its own inherent problems (among which the theory of irrationals, cf. above)[90]. Parts of Euclid's *Data* may

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[88] *Republica* 587d, ed., transl. SHOREY 1978: II, 396f. The implications of this passage are discussed in HØYRUP 1990c, text around note 7.

[89] I deal with these issues in HØYRUP 1990c.

[90] Already before the geometrical reinterpretation of Babylonian »algebra«, WILBUR KNORR (1975) proposed a connection between the »metric« geometry of *Element* II (etc.) and the techniques of calculators, more precisely the patterns of calculi (ψῆφοι) in figurate numbers etc. These patterns may, indeed, have much to do with what I have here called the »calculators' algebra«, see HØYRUP 1990c.

Several other interpretations of the origin of the techniques and propositions of represent a stage in this process which comes closer to the starting point: Are the sides of a rectangle really given when the area and the difference between/sum of its sides are given (Prop. 84−85[91])? When the area is given and the sides have a given ratio (Prop. 78[92])? When the area is given and the squares upon sides have a given ratio with excess (Prop. 86[93])? Or when it arises through the application of a given area to a given line directly or with excess or deficiency (Prop. 57−59[94])? A number of the ways in which magnitudes can be given according to the definitions[95] remind of Diophantos' *Arithmetica* I and of the Babylonian tablet BM 13 901 mentioned above: To be given in ratio, with excess or deficiency, or in ratio with excess or deficiency.

Part of Euclid's *Book on the Division of Figures,* on its part, seems to reflect a similar generalizing reflection of the cut-and-paste geometry of the surveyors' tradition. Prop. 33[96], in particular, which requires the partition of a trapezium by means of a parallel transversal in a given ratio, corresponds precisely to a clay tablet from the 23d century B.C.[97], with the only difference that this early partition is in ratio 1:1. This tablet constitutes the earliest positive trace of that Akkadian surveyors' tradition which seems to be behind the scribal »algebra« of the Old Babylonian era. As late as the 10th century, on the other hand, the Arabic mathematician Abūl-Wafā', tells about partitions and about the cut-and-paste predilections of practical geometers[98].

It thus appears that *all* Classical second-degree purported »algebra«, including the hotly disputed »geometrical algebra« (and even other

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Elements II have been proposed in recent years (FOWLER 1987; HERZ-FISCHLER 1987). I shall refrain from discussing whether these conjectures contradict the one suggested here or might serve as compatible complements.

[91] Ed., transl. MENGE 1896: 164-167.

[92] Ed., transl. MENGE 1896: 150-153. In arithmetical translation, we observe, this proposition comes close to Diophantos' trivial *Arithmetica* I,xxxi−xxxviii.

[93] Ed., transl. MENGE 1896: 168-173.

[94] Ed., transl. MENGE 1896: 102−109.

[95] Ed., transl. MENGE 1896: 2-5.

[96] Counted as in ARCHIBALD's reconstruction (1915: 72f).

[97] FRIBERG 1990: 541.

[98] *Book about that which is Necessary for Artisans in Geometrical Construction*, ed., transl. KRASNOVA 1966. See especially p. 115.

branches of scientific mathematics), grew out of or were inspired by the same sub-scientific soil[99]. No wonder that Hero, whose familiarity with calculators' second-degree algebra is demonstrated in various places, was able to give an »analytical«, i.e., quasi-algebraic, interpretation of *Elements* II[100].

VII. Other networks

The »Silk Road« family of arithmetical problems, the composite fractions, and the surveyors' and calculators' »algebras« were certainly not the only sub-scientific networks connecting the Hellenistic and Roman world with earlier, surrounding and later cultures. Much of Greco-Roman metrology was borrowed, as we know; the channel will have been contact with the original practitioners of the metrologies in question, i.e., mainly traders and surveyors. The adoption of the Egyptian unit fraction system and the amalgamation of this system with the Greek alphabetic numerals is a well-known phenomenon, which certainly took place first at the sub-scientific level. Good reasons could be given that another family of arithmetical problems found its way into Hellenistic culture that way[101]. Certain Archimedean results, most notably those connected to his

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[99] The one exception to this rule is the classification of irrationals and the study of the relations between classes in *Elements* X – actually the only piece of Ancient mathematics which relates in spirit to certain aspects of modern, »post-Noether« algebra. But for some obscure reason precisely this subject is normally left out from the search for Greek »algebra«.

[100] Se al-Nayrīzī's report of Hero's commentary to *Elements* II,1-10 in BESTHORN & HEIBERG (eds, transls) 1893: II, 4-61. MUELLER's discussion of the relation between Hero's single-line analysis and Euclid's two-dimensional proofs (1981: 46-50) is perspicacious; but if a »naive-geometric« interpretation is applied to Hero's »algebra«, it is no longer significantly different from MUELLER's alternative interpretation, »geometric assertions about the equality of certain areas useful for the transformation of one area or areas into another«.

[101] Most of the problems from *Anthologia graeca* XIV make use of »Greek«, i.e., Egyptian fractions. In Anania of Širak's collection of arithmetical problems, N° 22 (ed., transl. KOKIAN 1919: 116) deals with the distribution of wine to Pharaoh's officials at his birthday according to a scheme which is already familiar from Rhind Mathematical Papyrus. Since this collection owes much to Anania's stay in Byzantium, this familiarity with Egyptian mathematics has probably passed via Greece.

determination of the circular circumference and area, were adopted by practitioners and became part of the sub-scientific traditions[102]. Even certain definitely substandard (specifically Greco-Roman) practices appear to have been taken over in the bargain during the wholesale Western European appropriation of diluted Classical culture. One instance of this is the use of triangular numbers as measures of the area of the equilateral triangle, which was diffused together with the agrimensor writings and troubled the mathematically interested Adelbold around A.D. 998[103]. Another instance is the measurement of areas of figures by means of their circumference. In the Ancient sources it was not quite clear, as observed in note 13, what was meant by this, and how seriously it was meant. But in the *Propositiones* there is no doubt. In order to find out how many square *perticas* are contained within a circular field of circumference 400 *perticas*, the circumference is distributed as 4 times 100 perticas and the area then found as 100·100 (N° 25)[104], – we may presume that the field is transformed implicitly into a square to fit the square *perticas*. This is confirmed in N° 29, where rectangular houses of 30 feet times 20 feet are to be fitted into a circular city with circumference 8000 feet. This time, the circumference is explicitly divided as 4800 feet and 3200 feet along the length and width of the houses, respectively, and these then bisected and multiplied[105].

This questionable method was not reserved for recreational puzzles. In 1050, Franco of Liège tells (dissociating himself from the technique) that »there are also some who split the circular circumference in 4 parts, from

[102] Thus in the Hebrew *Mišnat ha-Middot*, which not only gives the value of the circular circumference as $3\frac{1}{7}$ of the diameter but also cites Rabbi Neḥemia (c. A.D. 150, and according to GANDZ the plausible author of the treatise) for the statement that this is what »the people of the world« (or, in another reading, »the landmeasurers«) say (ed., transl. GANDZ 1932: 49).

[103] In a letter to Gerbert, whose explanation was edited by BUBNOV 1899: 43. A more complete text is found in translation in LATTIN 1961: 299-301.

[104] Ed. FOLKERTS 1978: 59. 1 *pertica* equals 10 feet. Strictly speaking, what is asked for is the contents in square *aripenni*. Since 1 *aripennus* equals 120 feet, the resulting 10 000 are divided twice by 12.

[105] Ed. FOLKERTS 1978: 61. Certain manuscripts present a different solution to both problems, which happens to be numerically better but looks as a combination of disparate elements from Greek and Babylonian mensuration – cf. FOLKERTS 1978: 28.

which they span a square, claiming it to be equal to the circle«[106].

Summing up we may conclude that the indifference of the Classical sources toward basic mathematical technologies does not mean that these did not exist. Highly organized as it was on the administrative and commercial level, the Greco-Roman world could not do without them; and knowledge of the corresponding technologies used in geographically and temporally adjacent cultures allow us to extricate more information from the Classical sources than these would yield without supporting evidence. At the culturally subliminal level, the Classical world was traversed by a multitude of sub-scientific networks, more or less merged with each other.

We may also conclude that some of these technologies and networks were important for what went on at the culturally conscious level. Just as in the case of literature, the hidden undercurrents of non-literate and often oral culture provided an important part of the water and the nutrients which made literate scientific culture flourish.

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[106] *De quadratura circuli*, ed. FOLKERTS & SMEUR 1976: 65. My translation.

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